Class 19 – § 5.3 Properties of Logarithms

Basic property: $\boxed{b^y = x \iff \log_b x = y}$ $\quad b>0,\ b\neq 1$

$\log_b 1 = 0, \quad \log_b b = 1, \quad \cancel{b}^{(\cancel{\log_b} x)} = x, \quad \cancel{\log_b}(\cancel{b}^x) = x$

<u>Review</u>

1. Multiply binomials

<u>Ex1</u>: $(m+7)(m-9) = m^2 \underbrace{-9m+7m}\ -63 = m^2-2m-63.$

2. Solve quadratic equations with sqrt property

<u>Ex2</u>: $\sqrt{x^2} = \sqrt{19} \Rightarrow x = \pm\sqrt{19}$

<u>Ex3</u>: $(x-5)^2 - 16 = 0$ ($+16$ $+16$) $\Rightarrow \sqrt{(x-5)^2} = \sqrt{16} \Rightarrow x-5 = \pm 4$ ($+5$ $+5$) $\Rightarrow x = 5 \pm 4$

3. Radical equations $\sqrt[n]{x} = c$

<u>Ex4</u>: $(\sqrt[2]{x})^2 = (4)^2 \Rightarrow x = 16$

<u>Ex5</u>: $(\sqrt[4]{x})^4 = (4)^4 \Rightarrow x = 256$

If I raise both sides to an even exponent, I need to check the solution

4. Rational equations

<u>Ex6</u>: $\dfrac{4}{\cancel{x+8}}\cancel{(x+8)}(x-4) = \dfrac{2}{\cancel{x-4}}(x+8)\cancel{(x-4)}$ → Restricted values $\iff$ x values, den = 0
↳ 4 & −8

$4(x-4) = 2(x+8) \Rightarrow 4x - 16 = \cancel{2x} + 16$ ($-2x$ $-2x$) $\Rightarrow 2x \cancel{-16} = 16$ ($+16$ $+16$) $\Rightarrow \dfrac{2x}{2} = \dfrac{32}{2} \Rightarrow \boxed{x = 16}$

5. Evaluate Log expressions

$$\boxed{u^x = u^y \iff x = y}$$

Ex 7 : $\log_2 \frac{1}{64} = x \to$ base 2. Want: $2^x = \frac{1}{64} \Rightarrow 2^x = (2^6)^{-1} \Rightarrow 2^x = 2^{-6} \Rightarrow \boxed{x = -6}$

Ex 8 : $\log_{\sqrt{5}} 125 = x \to$ base 5. Want: $\sqrt{5}^x = 125 \Rightarrow (5^{\frac{1}{2}})^x = 5^3 \Rightarrow 5^{\frac{x}{2}} = 5^3$

$\Rightarrow \frac{x}{2} = 3 \Rightarrow \boxed{x = 6}$

Ex 9 : $\log 1000 = x \to$ base 10. $10^x = 1000 \iff 10^x = 10^3 \Rightarrow \boxed{x = 3}$

Ex 10 : $\log \frac{1}{10} = x \to$ base 10. $10^x = \frac{1}{10} \iff 10^x = 10^{-1} \Rightarrow \boxed{x = -1}$

Ex 11 : $\cancel{\ln e^2} = x \to$ base e. Want: $e^x = e^2 \Rightarrow \boxed{x = 2}$

§ 5.3 Properties of Logs

Let $b > 0$ & $b \neq 1$, u & v positive numbers, and r any number.

P1: Product Rule: $\log_b (u \cdot v) = \log_b u + \log_b v$

Ex: $\log_6 \underbrace{216}\underbrace{x} \overset{=}{\scriptstyle P1} \log_6 \underbrace{216}_{6^3} + \log_6 x = \cancel{\log_6} 6^3 + \log_6 x = 3 + \log_6 x$.

cancel

Never allowed: $\log_b (u+v) = \log_b u + \log_b v$ (WRONG)

P2: Quotient Rule: $\log_b \left(\frac{u}{v}\right) = \log_b u - \log_b v$

Ex: $\ln \left(\frac{e^{10}}{y}\right) \overset{=}{\scriptstyle P2} \cancel{\ln e}^{10} - \ln y = 10 - \ln y$

cancel

Never allowed: $\log_b (u-v) = \log_b u - \log_b v$ (WRONG)

OR : $\dfrac{\log_b u}{\log_b v} = \log_b u - \log_b v$ (WRONG)

P3: Power Rule: $\log_b(u^r) = r\cdot\log_b(u)$

Ex1: $\log_2(32) = \log_2(2^5) \underset{P3}{=} 5\cdot\underbrace{\log_2 2}_{1} = 5.$

Ex2: $\log_4(x^7) \underset{P3}{=} 7\cdot\log_4 x.$

Never allowed: $(\log_b u)^r = r\cdot\log_b u$ (WRONG)

Rewrite expressions

$2+\frac{1}{4} = \frac{8}{4}+\frac{1}{4} = \frac{9}{4}$

Ex1: Simplify $2\cdot\log_7 x + \frac{1}{4}\cdot\log_7 x$

$$2\log_7 x + \frac{1}{4}\log_7 x \underset{P3}{=} \log_7 x^2 + \log_7 x^{\frac{1}{4}} \underset{P1}{=} \log_7(x^2\cdot x^{\frac{1}{4}}) = \log_7(x^{\frac{9}{4}})$$

Ex2: Simplify $\log_2(x+8) + \log_2(x-8) \underset{P1}{=} \log_7[(x+8)(x-8)] = \log_7(x^2-64)$

Solving Log equations

$$\boxed{\log_b u = \log_b v \iff u = v}$$ ⊛

Ex1: $\log x = \log 100 \overset{⊛}{\iff} \boxed{x = 100}$

Ex2: $2\log_3 4 = \log_3 x \underset{P3}{\iff} \log_3 4^2 = \log_3 x \underset{⊛}{\iff} 4^2 = x$ or $\boxed{x=16}$

Ex3: $2\log(x+5) = \log 16 + \log 4 \underset{P1,P3}{\iff} \log(x+5)^2 = \log 16\cdot 4 \underset{⊛}{\iff} \sqrt{(x+5)^2} = \sqrt{64}$

$$\begin{array}{l} x+5 = \pm 8 \\ -5 \quad -5 \end{array} \iff x = -5\pm 8 \text{ or } x = \cancel{-13}, +3. \quad \boxed{x=3}$$

Remark: What goes inside log( ) must be positive.

Ex 4: $\log_9(x+8) - \log_9(x+3) = \log_9(x) \overset{P2}{\Longleftrightarrow} \log_9\left(\frac{x+8}{x+3}\right) = \log_9 x \overset{*}{\Longleftrightarrow}$

$\cancel{(x+3)}\underbrace{\frac{x+8}{\cancel{x+3}}}_{x=3 \text{ is restricted value}} = x\,(x+3) \Longleftrightarrow x+8 = x^2+3x \Longleftrightarrow 0 = x^2+2x-8$

$-x-8 \qquad -x-8$

$\Longleftrightarrow 0 = (x+4)(x-2) \Longleftrightarrow \cancel{x=-4}$ or $\boxed{x=2}$

Ex5: $\ln 2 + \boxed{\ln x} = \ln 4 + \boxed{\ln(2x+2)} \overset{P1}{\Longleftrightarrow} \ln(2x) = \ln\left(4(2x+2)\right) \overset{*}{\Longleftrightarrow}$

$2x = 4(2x+2) \Longleftrightarrow 2x = 8x+8 \Longleftrightarrow \frac{-6x}{-6} = \frac{8}{-6} \Longleftrightarrow \cancel{x = -\frac{4}{3}}$

$-8x \qquad -8x$

The problem has no solution.

Ex6: $\log_4(x+6) + \log_4(x-2) = \log_4(2x+3) \overset{P1}{\Longleftrightarrow} \log_4(x+6)(x-2) = \log_4(2x+3)$

$\overset{*}{\Longleftrightarrow} (x+6)(x-2) = 2x+3 \Longleftrightarrow x^2+4x-12 = 2x+3 \Longleftrightarrow x^2+2x-15=0$

$-2x-3 \qquad -2x-3$

$\Longleftrightarrow (x+5)(x-3)=0 \Longleftrightarrow \cancel{x=-5}$ or $\boxed{x=3.}$

Evaluate some logs

Change of base formula (CB)

$$\log_b u = \frac{\log_c u}{\log_c b}$$

Ex1: $\log_2 51 = \frac{\log 51}{\log 2}$ or $\frac{\ln 51}{\ln 2} \approx 5.6724$

$\log(51) \div \log(2)$ $\qquad$ $\ln(51) \div \ln(2)$

Ex2: $\log_{\frac{1}{7}} 83 = \frac{\log 83}{\log \frac{1}{7}}$ or $\frac{\ln 83}{\ln \frac{1}{7}} \approx -2.2708$

Ex3: $\log_4 x = \log_{16} 4 \Longleftrightarrow \log_4 x = \frac{\log_4 4}{\log_4 16} \Longleftrightarrow \log_4 x = \frac{1}{2} \Longleftrightarrow 4^{\frac{1}{2}} = x \Longleftrightarrow \boxed{x=2}$

Ex4: $\log_2 x = \log_4 25 \overset{CB}{\Longleftrightarrow} \log_2 x = \frac{\log_2 25}{\log_2 4} \Longleftrightarrow \log_2 x = \frac{\log_2 5^2}{\log_2 2^2} \overset{P3}{\Longleftrightarrow} \log_2 x = \frac{\cancel{2}\cdot\log_2 5}{\cancel{2}\cdot\underbrace{\log_2 2}_{1}}$

$\Longleftrightarrow \log_2 x = \log_2 5 \overset{*}{\Longleftrightarrow} \boxed{x=5}$